CONSTRUCTION OF NAVIER-STOKES EQUATION USING GAUGE FIELD THEORY APPROACH

A thesis submitted to the Fakultas Pasca-Sarjana Universitas Indonesia in partial fulfillment of the requirements for the degree of Master of Science
Graduate Program in Pure and Applied Physics

Universitas Indonesia

2005

Lembar Persetujuan

Judul Thesis: CONSTRUCTION OF NAVIER-STOKES EQUATION USING GAUGE FIELD THEORY APPROACH

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Sejatine ora ana apa-apa, sing ana iku dudu

(Pada hakekatnya tidak ada apa-apa, yang ada itu bukan)

'Sasangka Jati' bab panembah

Acknowledgements

I started working on my thesis when I joined a research cruise in the Indian Ocean with the Research Vessel MIRAI on July 2004. The objective of this cruise is to study Sea-Air interaction in the Eastern Equatorial Indian Ocean. During the time, I spent a lot on studying gauge field theory (It's very difficult subject that I have ever learned). I was inspired by Chandrasekhar when he got his theory of white dwarf star on his journey from India to England by the ships (he sailed in the Indian Ocean). Chandrasekhar is my hero and his picture is on my desk.

First of all, I would like to acknowledge my primary advisor, Dr. L.T. Handoko for all guidance, patience and critical comments. Before starting this work, I gave him a lecture about fluid dynamics but few months later he is getting to know fluid dynamics better than me. I can't believe it !!!

I am also indebted to Dr Terry Mart, he taught me about the Feynman's diagram, "the new thinking" for me. I will use it in the next research. I thank Dr. Anto Sulaksono, Dr. Imam Fachruddin and Dr. M. Hikam for their intriguing question during my defense. I would like to appreciate our theoretical group, Ketut Saputra, Ardy Mustafa, Freddy, Fahd, Jani, Ayung etc for so many valuable discussion.

The last but not the least, Sri Lestari and Dhita for their patience and I always love you all. My study has been supported by Pusat Pengkajian & Penerapan Teknologi Inventarisasi Sumberdaya Alam (P3-TISDA), Badan Pengkajian & Penerapan Teknologi (BPPT).

Jakarta, 2005

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Abstract

The equation of motion governs fluid flows is well known as the Navier-Stokes equation. Most researches on fluid dynamics are mostly dedicated to get the solutions of this equation with particular boundary conditions, because of difficulties in obtaining exact solutions for this kind of nonlinear equation. The gauge field theory is the most popular field theory and widely accepted as a basic theory in elementary particle physics. We then attempt to reconstruct the Navier-Stokes equation in the same manner as gauge theory. Using a four vector potential $\mathcal{A}_\mu$ with appropriate content describing the fluid dynamics, *i.e.*$A_\mu = (\Phi, \vec{A})$, we show that it is possible to construct the Navier-Stokes equation from a gauge invariant bosonic lagrangian $\mathcal{L}_{NS} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + g\mathcal{J}_\mu\mathcal{A}^\mu$. The Navier-Stoke equation is obtained as its equation of motion through the Euler-Lagrange equation.

Further, we present the application of the theory, *i.e.*the propagation Davydov soliton immersed in fluid system and the theory of turbulence. The propagation of Davidov soliton in fluid system that can be described by the Lagrange density which is similar to the quantum electrodynamics for boson particle. In the static condition, the Lagrange density is similar with the Ginzburg-Landau lagrangian. If fluid flow parallel to soliton propagation, the phenomenon is described by the variable that is a coefficient in the nonlinear Klein-Gordon equation. Behaviour of the solution in term of single solution is also given. Finally, concerning the similarity between the statistical mechanics and the fields theory we construct the theory of turbulence.

viii+30 pp.; appendices.

References: 35 (1961-2005)

Contents

Chapter 1

Introduction

Two of seven Millennium Prize Problems are:
5. The Yang-Mills Existence and Mass Gap
6. The Navier-Stokes Existence and Smoothness
It is a famous open question whether smooth initial
conditions always lead to smooth solutions for all times:
a $1,000,000$ *US dollar prize was offered in May 2000 by*
the Clay Mathematics Institute for the answer to these questions.
(http://encyclopedia.thefreedictionary.com/)

1.1 Background

The fluid dynamics still remains as an unsolved problem. Mathematically, a fluid flow is described by the Navier-Stokes (NS) equation [1]:

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{1}{\rho}\vec{\nabla}P - \mu\vec{\nabla}^2\vec{v} \,, \tag{1.1}$$

where $\vec{v}$ is fluid velocity, P is pressure, ρ is density and μ is the coefficient of viscosity. This equation is derived from the Newton's second law for fluid and is naturally nonlinear. This nonlinearity makes the system to be a complex and hard to handle because the lack of its simple superposed solutions. In a nonlinear system the solution does not obey a vector space and can not be superposed (added together) to produce new solutions. This makes it harder to solve than in a linear system.

In principle, the study of fluid dynamics is focused on solving the Navier-Stokes equation with particular boundary conditions. Mathematically it has been known as the boundary value problem. The most difficult problem in fluid dynamics is turbulence phenomenon. In the turbulence regime, the solution for the Navier-Stoke equation has a lot of Fourier modes, so that the solution is untrackable numerically or analytically. It is predicted that the strong turbulence has 10^{10} numerical operation [3]. We need another approach in fluid dynamics rather than the conventional one.

This thesis treats the fluid dynamics differently than the conventional point of view as seen in some fluid dynamics textbooks. In this approach, the fluid is described as a field of fluid buch. We use the gauge field theory to construct the fluid dynamics in a lagrangian. Objective of the research is to build a Langangian which can reproduce the Navier-Stoke equation as its equation of motion through the Euler-Lagrange principles by borrowing the gauge principle.

1.2 Overview

This thesis is organized as follow. The introduction and background of the problem are given in chapter one. Then a brief story of fluid dynamics will be described in chapter two. In the subsequent chapter we give a short review of gauge field theory. The main part is presented in chapter four. The discussion will be given in chapter five followed by summary.

Chapter 2

Fluid Dynamics

'We still do not understand how water flow'.
Richard P Feynman

In this chapter we describe the fluid flow briefly. Fluid dynamics is a branch of physics to study fluid (liquid or gases) flow. Fluid is a macroscopic phenomenon, that can be considered as a continuum medium. This implies that an element of fluid is small enough and can be treated as an infinitesimal. But it still contains a lot of molecules such that we can treat it as a macroscopic phenomenon. From this point of view, if we consider a fluid displacement, it is not the displacement of individual molecule but the displacement of a fluid element which contains a lot of molecule. The mathematical description of fluid flow obeys two description *i.e.*the Lagrange description and the Euler description. In the Lagrange description, fluid flow is described by a trajectory of fluid element. In the Euler description, fluid flow is described by a function of space and time. In this thesis we use the Euler description.

The fluid is characterized by two parameters, fluid velocity $\vec{v}(\vec{x}, t)$ and fluid density $\rho(\vec{x}, t)$ and the behavior of fluid flow obeys two laws, *i.e.*the conservation of mass and the conservation of momentum. A brief explanation of these laws is given below.

2.1 The Conservation of Mass

The conservation of mass means that fluid can be destroyed or be created. If we perturb fluid, the initial and final masses should remain the same. Let us consider a finite volume (V) of fluid with S is a closed surface of the finite volume (V). The mass of fluid in a finite volume is $\int\int\int \rho\, dV$. The mass of fluid flow through a closed surface is $\oint \rho dS$. The conservation of mass means that the incoming and outgoing flux of fluids are conserved per unit time in a finite volume (V). The statement can be written as:

$$\oint (\rho\vec{v}) \cdot dS = -\frac{\partial}{\partial t}\int \rho dV \tag{2.1}$$

Using the Gauss theorem, the left hand side becomes,

$$\begin{aligned} \int \vec{\nabla}\cdot(\rho\vec{v})dV &= -\frac{\partial}{\partial t}\int \rho dV \\ \int \left[\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v})\right] dV &= 0\ , \end{aligned} \tag{2.2}$$

and the result is,

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}(\rho\vec{v}) = 0\ . \tag{2.3}$$

This is called as the continuity equation.

2.2 The Conservation of Momentum

The conservation of momentum is just the Newton's second law. For a point particle with mass (m), then the law tells that $m\frac{d\vec{x}}{dt^2} = \vec{F}$, where $\vec{x}$ is the position of point particle. Expansion to fluid flow obeys $m \to \rho$ and the acceleration becomes $d^2\vec{x}/dt^2 \to D\vec{v}/Dt$ with $D/Dt = \partial/\partial t + \vec{v}\cdot\vec{\nabla}$ which is called as the material derivative.

The fundamental force in fluid flow is the stress gradient that can be written as follow,

$$F_i = -\frac{\partial}{\partial x_k}\Pi_{ik}\ , \tag{2.4}$$

where the stress tensor Π_{ik} is given by,

$$\Pi_{ik} = P\delta_{ik} - \sigma_{ik}\ , \tag{2.5}$$

where P is again pressure and σ_{ik} is viscosity tensor. The viscosity tensor is an asymmterics tensor generally. This tensor can be derived from molecular point of view through the transport Boltzmann equation. The viscosity tensor can be written as [4],

$$\sigma_{ij} = \mu \left(\frac{\partial U_i}{\partial x_k} + \frac{\partial U_k}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial U_l}{\partial x_l} \right) + \nu\delta_{ik}\frac{\partial U_l}{\partial x_l} \ , \tag{2.6}$$

where μ and ν are the kinematic and dynamic viscosity coefficients respectively. Substituting Eqs. (2.6), (2.5) and (2.4) into the Newton's second laws for fluid we get,

$$\rho \left(\frac{\partial \vec{v}}{\partial t} + (\vec{v}.\vec{\nabla})\vec{v} \right) = -\vec{\nabla}P + \mu\vec{\nabla}^2\vec{v} + \left(\nu + \frac{1}{3}\mu \right) \vec{\nabla}(\vec{\nabla} \cdot \vec{v}) \ . \tag{2.7}$$

This is called as the Navier-Stokes equation that governs the fluid dynamics.

Chapter 3

Gauge Field Theory

"The most incomprehensible thing about the world is that it is at all comprehensible "
Albert Einstein

Gauge field theory is a theory of field which is based on the gauge principles *i.e.*the theory is required to be invariant under a particular local gauge transformation. To illustrate this concept, let us consider a complex scalar field $\phi(x)$ in the Minkowski space-time. The lagrangian density of this field with potential V can be written as follow [5],

$$\mathcal{L}(\phi, \partial_\mu \phi) = (\partial^\mu \phi^*)(\partial_\mu \phi) - V(\phi^* \phi) \ . \tag{3.1}$$

If we impose a transformation,

$$\phi \to \phi' \equiv e^{-i\theta} \phi \ , \tag{3.2}$$

where θ is an arbitrary real constant, it's easy to show that the lagrangian density is invariant under this transformation. The transformation $e^{-i\theta}$ is called the global gauge transformation. Using Noether's theorem we have a conserved current (see for example [6]),

$$\mathcal{J}^\mu = \phi \partial^\mu \phi^* - \phi^* \partial^\mu \phi \ , \tag{3.3}$$

which satisfies,

$$\partial_\mu \mathcal{J}^\mu = 0 \ . \tag{3.4}$$

How about the local gauge transformation ? The local gauge transformation can be written as [5],

$$\phi \to \phi' \equiv e^{-i\theta(x)}\phi \; . \tag{3.5}$$

Under this transformation, the lagrangian density in Eq.(3.1) becomes,

$$\begin{aligned} \mathcal{L}(\phi, \partial_\mu \phi) &\to \mathcal{L}' \\ &= (\partial^\mu \phi^*)(\partial_\mu \phi) - V(\phi^* \phi) + (\partial^\mu \phi^*)(\partial_\mu \phi)(\partial_\mu \theta \partial^\mu \theta + \partial_\mu \theta - \partial^\mu \theta) \end{aligned} \tag{3.6}$$

which is clearly not invariant. In order to make the lagrangiant to be invariant, we must replace ∂^μ by a suitable transformation in the same manner as ϕ. Then, let us define a vector fields $\mathcal{A}^\mu(x)$ that is usually called 'gauge field' with the transformation rule [5],

$$\mathcal{A}^\mu \to \mathcal{A}' \equiv \mathcal{A}^\mu \;\; + \partial^\mu \theta \; , \tag{3.7}$$

and the covariant derivative is,

$$\mathcal{D}^\mu \equiv \partial^\mu \;\; + i\mathcal{A}^\mu \; . \tag{3.8}$$

Imposing the local gauge transformation,

$$\begin{aligned} \mathcal{D}^\mu \phi \to \left[\partial^\mu + i(\mathcal{A}^\mu + \partial^\mu)\right] e^{-i\theta}\phi &= e^{-i\theta}\partial^\mu \phi - ie^{-i\theta}\phi \partial^\mu \theta + ie^{-i\theta}\mathcal{A}^\mu \phi + ie^{-i\theta}\phi \partial^\mu \theta \\ &= e^{-i\theta}(\partial^\mu + i\mathcal{A}^\mu)\phi \\ &= e^{-i\theta}\mathcal{D}^\mu \phi \; , \end{aligned} \tag{3.9}$$

$$\mathcal{D}_\mu \phi^* \to e^{i\theta}\mathcal{D}_\mu \phi^* \; . \tag{3.10}$$

This shows how the covariant derivative is transformed in the same manner as ϕ. Replace ∂^μ with $\mathcal{D}^\mu$, therefore the lagrangian becomes,

$$\mathcal{L}(\phi, \mathcal{D}^\mu \phi) = (\mathcal{D}_\mu \phi^*)(\mathcal{D}^\mu \phi) - V(\phi^* \phi) \tag{3.11}$$

Now, our theory is invariant under a local gauge transformation, and we have a gauge field theory.

3.1 Abelian Gauge Field Theory

The lagrangian in Eq.(3.11) is not a closed dynamical system due to $\mathcal{A}^\mu$ which has newly been introduced an external field. To realize the boson field $\mathcal{A}^\mu$ to be a physical field, we must introduce the kinetic term for $\mathcal{A}^\mu$ but it should be invariant under the same transformation. This can be achieved by the form [7],

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \; , \tag{3.12}$$

with

$$F^{\mu\nu} = \partial^\mu \mathcal{A}^\nu - \partial^\nu \mathcal{A}^\mu \; . \tag{3.13}$$

Finally the total lagrangian becomes,

$$\mathcal{L}_A = (\mathcal{D}_\mu \phi^*)(\mathcal{D}^\mu \phi) - V(\phi^* \phi) - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \; . \tag{3.14}$$

This is the lagrangian (density) for a closed dynamical system that is invariant under a local gauge transformation.

This set of local gauge transformation forms $U(1)$ group. So the theory in Eq.(3.14) is called as the $U(1)$ gauge theory. The local gauge transformation can also be written as $\phi' = e^{-ig\theta(x)}\phi$ where g is a coupling constant. This kind of gauge theory is also called as abelian gauge theory since g forms a commutative algebra. Introducing g, then $\mathcal{A}^\mu$ transforms as,

$$\mathcal{A}^\mu \to \mathcal{A}' \equiv \mathcal{A}^\mu \; + g\partial^\mu \theta \; , \tag{3.15}$$

and the strength tensor in Eq.(3.13) transforms as,

$$\begin{aligned} F^{\mu\nu} \to F'^{\mu\nu} &= \partial^\mu (\mathcal{A}^\nu + ig\partial^\nu \theta) - \partial^\nu (\mathcal{A}^\mu + ig\partial^\mu) \\ &= \partial^\mu \mathcal{A}^\nu - \partial^\nu \mathcal{A}^\mu + ig\partial^\mu \partial^\nu \theta - ig\partial^\nu \partial^\mu \theta \\ &= \partial^\mu \mathcal{A}^\nu - \partial^\nu \mathcal{A}^\mu \\ &= F^{\mu\nu} \; , \end{aligned} \tag{3.16}$$

that is invariant.

The lagrangian density in Eq.(3.12) is still invariant under this local gauge transformation. An important relation between $\mathcal{D}_\mu$ and $F_{\mu\nu}$ is given by,

$$\begin{aligned}
[\mathcal{D}_\mu, \mathcal{D}_\nu] &= \mathcal{D}_\mu \mathcal{D}_\nu - \mathcal{D}_\nu \mathcal{D}_\mu \\
&= (\partial_\mu + i\mathcal{A}_\mu)(\partial_\nu + i\mathcal{A}_\nu) - (\partial_\nu + i\mathcal{A}_\nu)(\partial_\mu + i\mathcal{A}_\mu) \\
&= i\partial_\mu \mathcal{A}_\nu - i\partial_\nu \mathcal{A}_\mu + i^2 \mathcal{A}_\mu \mathcal{A}_\nu - i^2 \mathcal{A}_\nu \mathcal{A}_\mu \\
&= i(\partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu) + i^2 [\mathcal{A}_\mu, \mathcal{A}_\nu] \\
&= iF_{\mu\nu}
\end{aligned} \tag{3.17}$$

This relation can be obtained with a commutative relation $[\mathcal{A}_\mu, \mathcal{A}_\nu] = 0$. This relation can be used to prove a lagrangian is invariant under a local gauge transformation or not.

3.2 Non-Abelian Gauge Field Theory

We extend the algebra explained in the preceeding section to the non commutative (non abelian) algebra. The formalism can be used to describe a system of field (matter field) that generally contains multi-component field. The non abelian gauge transformation can be writen as [8],

$$U = e^{iT_a \theta(x)^a} \ , \tag{3.18}$$

where $T_a's$ is a set of matrices called as generator belongs to a particular Lie group and satisfy certain commutative relation $[T_a, T_b] = if_{abc}T_c$. f_{abc} is the structure constant that is completely anti-symmetric. The algebra satisfies this relation is known as Lie Algebra [7].

To get a non abelian field that is invariant under a local gauge tarnsformation, we must find similar relation to Eq. (3.17). This can be accomplished by introducing,

$$\mathcal{D}_\mu \equiv \partial_\mu \ + igT_a \mathcal{A}_\mu^a \ , \tag{3.19}$$

where g is again the gauge coupling constant. Then the commutative relation for

D_μ is,

$$\begin{aligned}
[\mathcal{D}_\mu, \mathcal{D}_\nu] &= \mathcal{D}_\mu \mathcal{D}_\nu - \mathcal{D}_\nu \mathcal{D}_\mu \\
&= (\partial_\mu + igT_a\mathcal{A}_\mu^a)(\partial_\nu + igT_a\mathcal{A}_\nu^a) - (\partial_\nu + igT_a\mathcal{A}_\nu^a)(\partial_\mu + igT_a\mathcal{A}_\mu^a) \\
&= igT_a(\partial_\mu\mathcal{A}_\nu^a - i\partial_\nu\mathcal{A}_\mu^a + i^2g^2T_a^2(\mathcal{A}_\mu^a\mathcal{A}_\nu^a - \mathcal{A}_\nu^a\mathcal{A}_\mu^a) \\
&= igT_a(\partial_\mu\mathcal{A}_\nu^a - \partial_\nu\mathcal{A}_\mu^a) + ig[\mathcal{A}_\mu^a, \mathcal{A}_\nu^a] \\
&= igT_aF_{\mu\nu}^a \ . \qquad (3.20)
\end{aligned}$$

The corresponding element of the Lie algebra $T_aF_{\mu\nu}^a = F_{\mu\nu}^a$ is given by,

$$F_{\mu\nu}^a = \partial^\mu\mathcal{A}_\nu^a - \partial_\nu\mathcal{A}_\mu^a + ig[\mathcal{A}_\mu^a, \mathcal{A}_\nu^a] \ , \qquad (3.21)$$

or

$$F_{\mu\nu}^a = \partial^\mu\mathcal{A}_\nu^a - \partial_\nu\mathcal{A}_\mu^a - gf^{abc}\mathcal{A}_\mu^b, \mathcal{A}_\nu^c \ . \qquad (3.22)$$

The commutative relation for a covariant derivative is,

$$[\mathcal{D}_\mu, \mathcal{D}_\nu] = igF_{\mu\nu}^a \ , \qquad (3.23)$$

where $F_{\mu\nu}^a$ is given by Eqs.(3.21) or (3.22). With this condition the lagrangian density becomes,

$$\mathcal{L} = -\frac{1}{4}F^{a\mu\nu}F_{\mu\nu}^a \ , \qquad (3.24)$$

which is invariant. The theory is called non abelian gauge theory or Yang-Mills field theory. The famous example of non abelian gauge theory is [8],

$$\mathcal{L}_{NA} = i\overline{\psi}\gamma^\mu(\partial_\mu\psi) - m_\psi\overline{\psi}\psi + gJ^{a\mu}A_\mu^a - \frac{1}{4}F_{\mu\nu}^aF^{a\mu\nu} \ . \qquad (3.25)$$

For instance, in case $a = 8$ it is well known as quantum chromodynamics (QCD). This theory is used to explain the strong interaction in hadron physics. The lagrangian of non abelian gauge theory include self-interaction among of gauge fields $\mathcal{A}_\mu^a$ through the term $gf^{abc}\mathcal{A}_\mu^b\mathcal{A}_n^cu$ in $F_{\mu\nu}^a$. The self-interaction is the main source of asymptotic freedom in quark chromodynamics [8].

Chapter 4

Navier-Stokes Equation from Gauge Field Theory

"We believe the unity of physics"

P.A.M Dirac

This chapter is the main part of this thesis. In this chapter we describe the Navier-Stokes equation using gauge field theory. This can be done through building the lagrangian in the similar manner as the preceeding chapter.

4.1 Maxwell-like equation for Ideal Fluids

The abelian gauge theory $U(1)$ is an electromagnetic theory that reproduces the Maxwell equation. To build a lagrangianan that is similar with a abelian gauge theory, we should 'derive' the Maxwell-like equation from the Navier-Stokes equation. The result can be used as a clue to construct a lagrangianan for fluid that satisfies gauge principle. Considering the Navier-Stokes equation Eq. (2.7) for an ideal fluid and incompressible condition,

$$\rho\left(\frac{\partial\vec{v}}{\partial t}+(\vec{v}.\vec{\nabla})\vec{v}\right)=-\vec{\nabla}P\ , \tag{4.1}$$

$$\vec{\nabla}\cdot\vec{v}=0\ . \tag{4.2}$$

Using the identity $\vec{v}\times(\vec{\nabla}\times\vec{v})=\vec{\nabla}(\frac{1}{2}\vec{v}^2)-(\vec{v}\cdot\vec{\nabla})\vec{v}$, the Navier-Stokes equation can be written as,

$$\frac{\partial\vec{v}}{\partial t}+\vec{\nabla}\left(\frac{1}{2}\vec{v}^2\right)-\vec{v}\times(\vec{\nabla}\times\vec{v})=-\frac{1}{\rho}\vec{\nabla}P\ , \tag{4.3}$$

and then,

$$\frac{\partial \vec{v}}{\partial t} = \vec{v} \times (\vec{\nabla} \times \vec{v}) - \vec{\nabla}\left(\frac{1}{2}\vec{v}^2 + \frac{P}{\rho}\right) . \tag{4.4}$$

Since the scalar potenstial $\Phi = \frac{1}{2}\vec{v}^2 + \frac{P}{\rho}$, the vorticity $\vec{\omega} = \vec{\nabla} \times \vec{v}$ and the Lamb's vector $\vec{l} = \vec{\omega} \times \vec{v}$, the equation becomes,

$$\begin{aligned} \frac{\partial \vec{v}}{\partial t} &= -\vec{\omega} \times \vec{v} - \vec{\nabla}\Phi \\ &= -\vec{l} - \vec{\nabla}\Phi . \end{aligned} \tag{4.5}$$

Imposing curl operation in Eq. (4.5) we get the vorticity equation as follow,

$$\frac{\partial \vec{\omega}}{\partial t} = -\vec{\nabla} \times (\vec{\omega} \times \vec{v}) . \tag{4.6}$$

To get the Maxwell-like equation for an ideal fluid, let us take divergence operation for Eq. (4.5),

$$\begin{aligned} \frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{v}) &= -\vec{\nabla} \cdot \vec{l} - \vec{\nabla}^2\Phi \\ \vec{\nabla} \cdot \vec{l} = -\vec{\nabla}^2\Phi &= \tilde{\rho} \end{aligned} \tag{4.7}$$

In this result we have used an incompressible condition. The divergence of a vorticity is always zero (by definition of the vorticity), *i.e.*

$$\vec{\nabla} \cdot \vec{\omega} = 0 . \tag{4.8}$$

Again imposing curl operation, we have:

$$\begin{aligned} \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{v}) &= -\vec{\nabla} \times \vec{l} - \vec{\nabla} \times (\vec{\nabla}\Phi) , \\ \frac{\partial \vec{\omega}}{\partial t} &= -\vec{\nabla} \times \vec{l} , \\ \vec{\nabla} \times \vec{l} &= -\frac{\partial \vec{\omega}}{\partial t} , \end{aligned} \tag{4.9}$$

where we have used the identity $\vec{\nabla} \times (\vec{\nabla} \cdot \phi) = 0$.

Now, let us consider the definition of the Lamb's vector $\vec{l} = \vec{\omega} \times \vec{v}$. Taking the derivative $\partial/\partial t$ in the definition we obtain,

$$\frac{\partial \vec{l}}{\partial t} = \frac{\partial \vec{\omega}}{\partial t} \times \vec{v} + \vec{\omega} \times \frac{\partial \vec{v}}{\partial t} . \tag{4.10}$$

Substituting Eq. (4.5) and (4.6),

$$\frac{\partial \vec{l}}{\partial t} = \vec{v}^2(\vec{\nabla} \times \vec{\omega}) - \vec{j} \ , \tag{4.11}$$

or,

$$\vec{\nabla} \times \vec{\omega} = \alpha \vec{j} + \alpha \frac{\partial \vec{l}}{\partial t} \ , \tag{4.12}$$

where,

$$\alpha = \frac{1}{\vec{v}^2} \ , \tag{4.13}$$

$$\vec{j} = -\vec{v}\vec{\nabla}^2\Phi + \left[\vec{\nabla} \times (\vec{v} \cdot \vec{\omega})\right] \vec{v} + \vec{\omega} \times \vec{\nabla}(\Phi + \vec{v}^2) + 2\left[(\vec{\nabla} \times \vec{v}) \cdot \vec{\nabla}\right] \vec{v} \ . \tag{4.14}$$

So, we have Maxwell-like equation for fluids as,

$$\vec{\nabla} \cdot \vec{l} = \tilde{\rho} \ , \tag{4.15}$$

$$\vec{\nabla} \times \vec{l} = -\frac{\partial \vec{\omega}}{\partial t} \ , \tag{4.16}$$

$$\vec{\nabla} \cdot \vec{\omega} = 0 \ , \tag{4.17}$$

$$\vec{\nabla} \times \vec{\omega} = \alpha \vec{j} + \alpha \frac{\partial \vec{l}}{\partial t} \ . \tag{4.18}$$

Analogue to the electromagnetic field, we have $\vec{l}$ corresponds to $\vec{E}$ and $\vec{\omega}$ corresponds to $\vec{B}$. Conventionally, the vector $\vec{E}$ and $\vec{B}$ can be written in term of scalar potential ϕ and vector potential $\vec{A}$ as, $\vec{E} = -\vec{A}/\partial t - \vec{\nabla}\phi$ and $\vec{B} = \vec{\nabla} \times \vec{A}$. Therefore, it implies that $\vec{A}$ should correspond to $\vec{v}$. Using Eq. (4.5), we have,

$$\vec{l} = -\frac{\vec{v}}{\partial t} - \vec{\nabla}\Phi \ , \tag{4.19}$$

such that ϕ corresponds to Φ. If the fluid velocity is time independent, then $\vec{l} = -\vec{\nabla}\Phi$. This is the "electrostatic" condition.

We use these results to develop gauge field theory approach for fluid dynamics in the next section.

4.2 Minkowski Space-Time Formulation

In the Minkowski space-time formulation, the diagonal metric tensor has elements $g^{00} = 1$, $g^{11} = g^{22} = g^{33} = -1$. Now,we define a four vector $\mathcal{A}_\mu$ as follow:

$$\mathcal{A}_\mu = (A_o, \vec{A}) = (\Phi, -\vec{v}) \tag{4.20}$$

where $\Phi = \frac{1}{2}\vec{v}^2 + V$, with V is a potential induced by conservative force. Further we define the strength tensor as,

$$\mathcal{F}_{\mu\nu} \equiv \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu \ , \tag{4.21}$$

Now we construct a lagrangian for fluid system. Fluid can be viewed as a gauge boson that similar to gauge theory $U(1)$. The Lagrangian for fluid can be written as,

$$\mathcal{L}_{NS} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + g \mathcal{J}_\mu \mathcal{A}^\mu \tag{4.22}$$

where $\mathcal{J}_\mu$ is just a four-vector currents. To get the equation of motion we use the Euler-lagrange equation,

$$\partial^\nu \frac{\partial \mathcal{L}_{NS}}{\partial(\partial^\nu \mathcal{A}^\mu)} - \frac{\partial \mathcal{L}_{NS}}{\partial \mathcal{A}^\mu} = 0 \ . \tag{4.23}$$

After a straightforward calculation, the second term gives,

$$\frac{\partial \mathcal{L}_{NS}}{\partial \mathcal{A}^\mu} = g \mathcal{J}_\mu \ . \tag{4.24}$$

For calculating the first term, we write the langangian explicitely in term of A_μ,

$$\mathcal{L}_{NS} = -\frac{1}{4}(g_{\lambda\alpha})(g_{\beta\sigma})[(\partial^\alpha \mathcal{A}^\sigma - \partial^\sigma \mathcal{A}^\alpha)(\partial^\lambda \mathcal{A}^\beta - \partial^\beta \mathcal{A}^\lambda)] + g \mathcal{J}_\mu \mathcal{A}^\mu \ . \tag{4.25}$$

Substituting it into the first term in Eq.(4.23),

$$\begin{aligned}
\frac{\partial \mathcal{L}_{NS}}{\partial(\partial^\nu \mathcal{A}^\mu)} &= -\frac{1}{4}(g_{\lambda\alpha})(g_{\beta\sigma})\frac{\partial}{\partial(\partial^\nu \mathcal{A}^\mu)}[(\partial^\alpha \mathcal{A}^\sigma - \partial^\sigma \mathcal{A}^\alpha)(\partial^\lambda \mathcal{A}^\beta - \partial^\beta \mathcal{A}^\lambda)] \\
&= -\frac{1}{4}(g_{\lambda\alpha})(g_{\beta\sigma})[\frac{\partial(\partial^\alpha \mathcal{A}^\sigma)}{\partial(\partial^\nu \mathcal{A}^\mu)}\mathcal{F}^{\lambda\beta} - \frac{\partial(\partial^\sigma \mathcal{A}^\alpha)}{\partial(\partial^\nu \mathcal{A}^\mu)}\mathcal{F}^{\lambda\beta} \\
&\quad + \mathcal{F}^{\alpha\sigma}\frac{\partial(\partial^\lambda \mathcal{A}^\beta)}{\partial(\partial^\nu \mathcal{A}^\mu)} - \mathcal{F}^{\alpha\sigma}\frac{\partial(\partial^\beta \mathcal{A}^\lambda)}{\partial(\partial^\nu \mathcal{A}^\mu)} \\
&= -\frac{1}{4}(g_{\lambda\alpha})(g_{\beta\sigma})[\delta^\alpha_\nu \delta^\sigma_\mu \mathcal{F}^{\lambda\beta} - \delta^\sigma_\nu \delta^\alpha_\mu \mathcal{F}^{\lambda\beta} + \delta^\lambda_\nu \delta^\beta_\mu \mathcal{F}^{\alpha\sigma} - \delta^\beta_\nu \delta^\lambda_\mu \mathcal{F}^{\alpha\sigma}]
\end{aligned} \tag{4.26}$$

Due to the symmetry of $g_{\mu\nu}$ and anti-symmetry of $\mathcal{F}_{\mu\nu}$, all four terms are equal,

$$\frac{\partial \mathcal{L}_{NS}}{\partial(\partial^\nu \mathcal{A}^\mu)} = -\frac{1}{4}[\mathcal{F}^{\nu\mu} - (-\mathcal{F}^{\nu\mu}) + \mathcal{F}^{\nu\mu} - (\mathcal{F}^{\nu\mu})] = -\frac{1}{4}(4\mathcal{F}^{\nu\mu}) = \mathcal{F}_{\mu\nu} \ . \tag{4.27}$$

Then the Euler-Lagrange equation becomes,

$$\begin{aligned}
0 &= \partial^\nu \mathcal{F}_{\mu\nu} - g\mathcal{J}_\mu \\
&= \partial^\nu(\partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu) - g\mathcal{J}_\mu \\
&= \partial^\nu \partial_\mu \mathcal{A}_\nu - \partial^\nu \partial_\nu \mathcal{A}_\mu) - g\mathcal{J}_\mu \ .
\end{aligned} \tag{4.28}$$

Now integrating it over x^ν,

$$\partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu = g \oint \mathrm{d}x_\nu \mathcal{J}_\mu \ . \tag{4.29}$$

For $\nu = \mu$ we obtain a trivial relation. The non trivial relation is obtained for $\nu \neq \mu$. Calculation of its components gives,

$$\partial_0 A_i - \partial_i A_0 = -g \oint \mathrm{d}x_0 J_i = g \oint \mathrm{d}x_i J_0 \ . \tag{4.30}$$

Since $A_i = -\vec{v}$, $A_o = \Phi$, $\partial_o = \partial/\partial t$ and $\partial_i = \vec{\nabla}$. we have,

$$-\frac{\partial \vec{v}}{\partial t} - \vec{\nabla}\Phi = -g\vec{\tilde{J}} \ , \tag{4.31}$$

where $\tilde{J}_i \equiv \oint \mathrm{d}x_0 J_i = -\oint \mathrm{d}x_i J_0$. Concerning the scalar potential given by $\Phi = \frac{1}{2}\vec{v}^2 + V$, we obtain,

$$-\frac{\partial \vec{v}}{\partial t} - \frac{1}{2}\vec{\nabla}\,|\vec{v}|^2 - \vec{\nabla}V = -g\vec{\tilde{J}} \ . \tag{4.32}$$

Borrowing the identity $\frac{1}{2}\vec{\nabla}\,|\vec{v}|^2 = (\vec{v}\cdot\vec{\nabla})\vec{v} + \vec{v}\times(\vec{\nabla}\times\vec{v})$, we get,

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\vec{\nabla}V - \vec{v}\times\vec{\omega} - g\vec{\tilde{J}} \ , \tag{4.33}$$

where $\vec{\omega} \equiv \vec{\nabla}\times\vec{v}$ is the vorticity. This result reproduces the general NS equation with arbitrary conservative forces $(\vec{\nabla}V)$.

The potential V can be associated with some known forces, for example,

$$V = \begin{cases} \dfrac{P}{\rho} & : \quad \text{pressure} \\ \dfrac{Gm}{r} & : \quad \text{gravitation} \\ (\nu+\eta)(\vec{\nabla}\cdot\vec{v}) & : \quad \text{viscosity} \end{cases} \tag{4.34}$$

Here, $P, \rho, G, \nu+\eta$ denote pressure, density, gravitational constant and viscosity as well. However we should put an attention on the potential of viscosity. We can extract a general force of viscosity $\vec{\nabla}V_{\text{viscosity}} = \eta\vec{\nabla}\left(\vec{\nabla}\cdot\vec{v}\right) + \nu\left(\vec{\nabla}^2\vec{v}\right) + \nu\left(\vec{\nabla}\times\vec{\omega}\right)$ using the identity $\vec{\nabla}\times\vec{\omega} = \vec{\nabla}(\vec{\nabla}\cdot\vec{v}) - \vec{\nabla}^2\vec{v}$. This reproduces both compressible and incompressible fluids, while contributes to the turbulence fluid for non-zero $\vec{\omega}$.

From Eq. (4.29) we can write explicitly the expression for the 4-vector current as,

$$J_0 = \rho \ = \ \frac{1}{g}\vec{\nabla}\cdot\left(\partial_0\vec{A} - \vec{\nabla}\phi\right) \ , \tag{4.35}$$

$$\vec{J} \ = \ -\frac{1}{g}\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right) - \partial^0\left(\partial_0\vec{A} - \vec{\nabla}\phi\right) \ , \tag{4.36}$$

Taking the time derivative operation to Eq. (4.35) and divergence operation to Eq. (4.38) we get,

$$\frac{\partial \rho}{\partial t} = \frac{1}{g}\left[\partial_t^2(\vec{\nabla}\cdot\vec{A}) - \vec{\nabla}^2(\partial_t\phi)\right] , \tag{4.37}$$

$$\vec{\nabla}\cdot\vec{J} = -\frac{1}{g}\vec{\nabla}\cdot\left[\vec{\nabla}\times(\vec{\nabla}\times\vec{A})\right] - \frac{1}{g}\left[\partial_t^2(\vec{\nabla}\cdot\vec{A}) - \vec{\nabla}^2(\partial_t\phi)\right] , \tag{4.38}$$

or in the four-vector formalism it can be written as $\partial_\mu \mathcal{J}^\mu = 0$. Using vector identity $\vec{\nabla}\cdot\vec{\nabla}\times\vec{a} = 0$,

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0 , \tag{4.39}$$

that is the continuity equation.

So far we obtain two fundamental equations for fluid dynamics. In the lagrangian, g is expected to be small contant coupling $(g\langle\langle 1)$. With this fact we can use a perturbation method of field theory to perform any calculation in fluid dynamics starting from the lagrangianan Eq.(4.22). For multi-fluids system we can expand it trivially,

$$\mathcal{L}_{NS} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + g\mathcal{J}^a_\mu\mathcal{A}^{a\mu} . \tag{4.40}$$

This is similar to the nonabelian gauge theory.

4.3 Euclidean Space-Time Formulation

The Navier-Stokes equation is a classical one described in the Euclidean space-time. It is useful to describe our formulation in Euclidean space-time (4-space). In principles, Euclidean 4-space can be obtained from Minkowski 4-space by clockwising the real axis in the complex x_o plane into the negative imaginary axis [5]. If a position in Euclidean 4-space is denoted by $x_E = (x_0, \vec{x})$, where x_0 is a real parameter. Then the relation with the Minkowski 4-space $x_\mu = (x_0, -\vec{x})$ is given by $x_0 \to ix_0$. Using this replacement many relations hold, as

$$\begin{aligned} x_E^2 &= x_0^2 + x_1^2 + x_2^2 + x_3^2 \\ &\to (ix_0)^2 + x_1^2 + x_2^2 + x_3^2 \\ &= (t^2 - \vec{x}\cdot\vec{x}) = -x^\mu x_\mu , \end{aligned} \tag{4.41}$$

$$d^4x = -id^4x_E . \tag{4.42}$$

The differential operator is then given by,

$$\partial_\mu = \frac{\partial}{\partial x^\mu} \to \frac{\partial}{\partial x_E} = \left(i\frac{\partial}{\partial x_0}, \vec{\nabla} \right) . \tag{4.43}$$

For Euclidean vector, there is no difference between upper and lower indices. Thus in the equation of motion we just replace x^o by ix_0 and also the associated differential operator. The Lorentz invariance of the lagrangian density is then replaced by the invariance under $O(4)$ rotation in Euclidean 4-space.

The gauge boson $\mathcal{A}_\mu$ will be replaced by $\mathcal{A}_\mu(x_E)$ with real component according to $A_i \to A_i(x_E)$ and $A_o \to iA_0(x_E)$. For example it turns the divergence of $\mathcal{A}_\mu$ out to be,

$$\partial_\mu \mathcal{A}^\mu = \frac{\partial \mathcal{A}^\mu}{\partial x^\mu} \to \frac{\partial \mathcal{A}^\mu}{\partial x^\mu} = \left(i\partial x_0, \vec{\nabla}\right) \cdot \left(iA_0, \vec{A}\right) = -\left(\frac{\partial A_0}{\partial x_0} - \vec{\nabla} \cdot \vec{A}\right) . \tag{4.44}$$

Now, we reconstruct all previous formulations in the Euclidean 4-space. The equation of motion in Eq. (4.29) reads,

$$\frac{\partial \mathcal{A}_\nu(x_E)}{\partial x_E^\mu} - \frac{\partial \mathcal{A}_\mu(x_E)}{\partial x_E^\nu} = -ig \oint \mathrm{d}^4 x_E \, .\mathcal{J}_\mu(x_E) \tag{4.45}$$

As before, the non trivial solution is obtained for $\mu \neq \nu$,

$$i\frac{\partial \mathcal{A}_i}{\partial x^0} - i\frac{\partial \mathcal{A}_0}{\partial x^i} = -ig \oint \mathrm{d}x_0 \mathcal{J}_i \, . \tag{4.46}$$

Using $A_0 = \Phi$, $A_i = -u_i$ and $x^0 = t$ we arrive at,

$$-\frac{\partial \vec{v}}{\partial t} - \frac{1}{2}\vec{\nabla}\, |\vec{v}|^2 - \vec{\nabla} V = -g\vec{\tilde{J}} \, . \tag{4.47}$$

Borrowing the identity $\frac{1}{2}\vec{\nabla}\, |\vec{v}|^2 = (\vec{v} \cdot \vec{\nabla})\vec{v} + \vec{v} \times (\vec{\nabla} \times \vec{v})$, the result is,

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\vec{\nabla} V - \vec{v} \times \vec{\omega} - g\vec{\tilde{J}} \, . \tag{4.48}$$

This is actually the Navier-Stoke equation as derived in the Minkowski space, Eq. (4.33).

Chapter 5

Discussion

A theory has only alternative of being right or wrong.
A model has a third possibility: it may be right, but irrelevant.
Manfred Eigen

5.1 Equation of Motion

Fluid dynamics is described by nonlinear partial differential equations called as Navier-Stokes equation. Exact solutions are available just for few simple situations. However, for the rest majority of situations, explicit solutions are not readily available. Moreover, it is not in general possible to solve the full Navier-Stokes equation if the flow is turbulence. On the other hand, these types of flow are the most common in nature. In the turbulence regime, the flow is characterized by chaotic unpredictable change in space and time due to appearance of fluctuations with arbitrarily small scale [9]. In this section we review field theory approach to fluid dynamics that have been done by many researchers and then we compare them with our methods.

The work has been pionereed with the similarities between the equation of fluid dynamics and those of quantum mechanics by Madelung more than 70 years ago [15, 16, 17]. In these approaches, the lagrangian for the Navier-Stokes equation has been constructed from the non-relativistic field theory. Starting from Schrodinger equation, the lagrangian has been found to be [17],

$$\mathcal{L}_S = \int d^d r \{ i\psi^* \dot{\psi} - \frac{1}{2}\nabla\psi^* \cdot \nabla\psi - \tilde{V}(\psi^*\psi) \ , \tag{5.1}$$

where "dot" denotes time derivative. Imposing a transformation as,

$$\psi = \rho^{\frac{1}{2}} e^{i\theta} \ , \tag{5.2}$$

where ρ and θ are real. We can obtain the new lagrangian as,

$$\mathcal{L}_{NS} = \int d^d r \{ -\rho\dot{\theta} - \frac{1}{2}(\nabla\theta)^2 - V(\rho)\} \ , \tag{5.3}$$

where $V(\rho) = \tilde{V}(\rho) + \frac{1}{8\rho}(\nabla\theta)^2$. Substituting this into the Euler-lagrangian equation we get,

$$\frac{\partial\rho}{\partial t} + \nabla \cdot (\rho\nabla\theta) = 0 \ , \tag{5.4}$$

$$\frac{\partial\theta}{\partial t} + \frac{1}{2}(\nabla\theta)^2 - \frac{\delta V'}{\delta\rho} = 0 \ , \tag{5.5}$$

where $V' = \int d^d r V(\rho)$. The equation Eqs. (5.4) and (5.5) are called the continuity and Navier-Stokes equation respectively. The transformation in Eq.(5.2) is usually used to study relationship between the nonlinear Schrodinger equation and the hydrodynamics phenomenon such as difusivity, Chern-Simon hydrodynamics and fractal hydrodynamics [16, 18, 19] .

The Lagrangian formulation for non-abelian fluid dynamics was proposed by Jackiw which has been applied to the quark-gluon plasma [20, 21]. In this formalism, the lagrangian density can be written as [21],

$$\mathcal{L}_{NS} = -j^\mu a_\mu + \frac{1}{2}\rho \overrightarrow{v}^2 - V \ , \tag{5.6}$$

where $x^\mu = (ct, x, y, z)$, $j^\mu = (c\rho, \rho\overrightarrow{v})$, $a_\mu \equiv \partial_\mu\theta + \alpha\partial_\mu\beta$ and $\overrightarrow{v} = \nabla\theta + \alpha\nabla\beta$. Again using the Euler-lagrangian equation in term of ρ we get the Bernoulli equation,

$$\frac{\partial\theta}{\partial t} + \alpha\frac{\partial\beta}{\partial t} + \frac{1}{2}(\overrightarrow{v})^2 + \frac{\delta\int V dr}{\delta\rho} = 0 \ . \tag{5.7}$$

Taking the gradient we then obtain the Navier-Stokes equation.

Another development is the application of gauge principles in the flow of an ideal fluid proposed by T. Kombe [22]. A free-field lagrangian is defined with a constraint condition of continuity equation and invariance againts the global $SO(3)$ gauge transformation. The lagrangian density is given by,

$$\mathcal{L}_{NS} = \int d^3x \{ \frac{1}{2}\rho\vec{v}^2 - \rho\varepsilon(\rho) + \phi\frac{\partial\rho}{\partial t} + \phi\nabla\cdot(\rho\vec{v})\} \ , \tag{5.8}$$

where ϕ a scalar function as a Langrange multiplier and ε the internal energy per unit mass. The Euler-lagrange equations have been derived by varying ϕ, $\vec{v}$ and ρ. Using gradient operation we get the continuity equation and the Navier-Stokes equation respectively,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \ , \tag{5.9}$$

$$\frac{\partial \vec{v}}{\partial t} + \nabla(\frac{1}{2}\vec{v}^2) = -\nabla h \ , \tag{5.10}$$

where $h = \varepsilon + \rho \frac{d\varepsilon}{d\rho}$. This formulation works only for irrotational flow since $curl \vec{v} = 0$. Using "gauge principles",

$$\vec{v} \rightarrow \vec{v}' = e^{\theta} \vec{v} \ , \tag{5.11}$$

$$\nabla_t \vec{v} \rightarrow \nabla'_t \vec{v}' = e^{\theta} \nabla_t \vec{v} \ , \tag{5.12}$$

the Navier-Stokes equation has been obtained,

$$\frac{\partial \vec{v}}{\partial t} + (\nabla \times \vec{v}) \times \vec{v} + \nabla(\frac{1}{2}\vec{v}^2) = -\nabla h \ , \tag{5.13}$$

Some authors use gauge principle in the Hamiltonian formalism to produce the fluid dynamics equation [23, 24]. From this point of view, the dynamics of hydrodynamics system is described in the phase space of field and is determined by the complete set of the Poisson brackets. We will not discuss of their formalisms further.

In this thesis, our approach is difference. We started with the similarity between the Electromagnetism and Fluid dynamics. As done in Chap. 4, we have constructed the Maxwell-like equation for an ideal fluid,

$$\vec{\nabla} \cdot \vec{l} = \tilde{\rho} \ , \tag{5.14}$$

$$\vec{\nabla} \times \vec{l} = -\frac{\partial \vec{\omega}}{\partial t} \ , \tag{5.15}$$

$$\vec{\nabla} \cdot \vec{\omega} = 0 \ , \tag{5.16}$$

$$\vec{\nabla} \times \vec{\omega} = \alpha \vec{j} + \alpha \frac{\partial \vec{l}}{\partial t} \ . \tag{5.17}$$

The correspondences of the electromagnetism and the ideal fluid can be written as follow,

$$\begin{array}{rcl} \overrightarrow{B} & \leftrightarrow & \overrightarrow{\omega}\ , \\ \overrightarrow{E} & \leftrightarrow & \overrightarrow{l}\ , \\ \overrightarrow{A} & \leftrightarrow & \overrightarrow{v}\ , \\ \phi & \leftrightarrow & \Phi\ , \end{array} \tag{5.18}$$

where $\overrightarrow{B}$ is the magnetic field, $\overrightarrow{E}$ is the electric field, $\overrightarrow{A}$ is the electromagnetics vector, ϕ is a scalar function, $\overrightarrow{\omega}$ is the fluid vorticity, $\overrightarrow{l}$ is the Lamb's vector, $\vec{v}$ is fluid velocity and Φ is the scalar potential. The same as the electromagnetics field, we have a four vector $A_\mu = (\phi, \vec{A})$ which can be constructed to be the four vector for fluid dynamics, $\mathcal{A} = (\Phi, \vec{v})$. In the electromagnetics field, the scalar potential ϕ and the vector $\vec{A}$ is an auxiliary field, but in the fluid dynamics the scalar potential $\Phi = \frac{1}{2}\vec{v}^2 + V$ describes energy of fluid, while the vector $\vec{v}$ is fluid velocity (a physical observable). Similar to the electromagnetics field, we construct the lagrangian density,

$$\mathcal{L}_{NS} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + g \mathcal{J}_\mu \mathcal{A}^\mu\ , \tag{5.19}$$

where,

$$\mathcal{F}_{\mu\nu} \equiv \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu\ . \tag{5.20}$$

Using the Euler-lagrangian equation we obtain the Navier-Stokes and the continuity equation respectively,

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\vec{\nabla} V - \vec{v} \times \vec{\omega} - g\vec{\tilde{J}}\ , \tag{5.21}$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0 \tag{5.22}$$

Similar to the electromagnetics theory, we can define the energy-momentum tensor as,

$$T_{\mu\nu} = \partial^\mu A_\lambda \frac{\partial \mathcal{L}_{NS}}{\partial(\partial^\nu A_\lambda)} - g_{\mu\nu} \mathcal{L}_{NS}\ . \tag{5.23}$$

This is not a symmetric tensor. The symmetric tensor can be obtained by subtracting a term involving the sum $\partial^\lambda A_\mu \mathcal{F}_{\lambda\nu}$, that is,

$$\Theta_{\mu\nu} = T_{\mu\nu} - (\partial^\lambda A_\mu)\mathcal{F}_{\lambda\nu} = -\mathcal{F}_{\lambda\mu}\mathcal{F}_{\lambda\nu} - g_{\mu\nu}\mathcal{F}_{\alpha\beta}\mathcal{F}^{\alpha\beta}\ . \tag{5.24}$$

For example, if we use "free" Lagrangian $\mathcal{L}_{NS} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, an explicit calculation gives the following component,

$$\begin{aligned}
\Theta_{00} &= -\mathcal{F}_{\lambda 0}\mathcal{F}_{\lambda 0} - \frac{1}{4}\mathcal{F}_{\alpha\beta}\mathcal{F}^{\alpha\beta} \\
&= l^2 + \frac{\omega^2 - l^2}{2} \\
&= \frac{1}{2}(\omega^2 + l^2) \ ,
\end{aligned} \tag{5.25}$$

where $\omega = |\nabla \times \vec{v}|$ and $l = |\overrightarrow{\omega} \times \vec{v}|$. These are called as vorticity and Lamb's vector respectively. Another components are given by,

$$\Theta_{0i} = (\overrightarrow{l} \times \overrightarrow{\omega})_i \ , \tag{5.26}$$

and

$$\Theta_{ij} = -\left[l_i l_j + \omega_i \omega_j - \frac{1}{2}\delta_{ij}(l^2 + \omega^2) \right] \ . \tag{5.27}$$

5.2 Application of the Theory

In this section we describe an idea to apply the theory. The first topic is ***interaction between soliton and fluid system***. Soliton is a pulse-like nonlinear wave which forms a collision with similar pulse having unchanged shape and speed [25]. The wave equations that exhibit soliton are the KdV equation, the Nonlinear Schrodinger equation, the Sine-Gordon equation, the Born-Infeld equation, the Burger equation and the Boussiness equation. We only focus on the Sine-Gordon equation.

The Sine-Gordon equation appears in many area of physics. For example the behavior of muscle contraction [26], one-dimension easy-plane ferromagnetics [27], the self-induced transparency that describes the traveling of ultrashort pulses of light though a resonant two-level optical medium [25] and the dynamics of α helical protein [28].

Consider the lagrangian density,

$$\mathcal{L} = \frac{1}{2}\phi_t^2 - \frac{1}{2}\phi_x^2 + cos\phi \ . \tag{5.28}$$

Using the Euler-lagrangian equation, we get the Sine-Gordon equation,

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} + sin\phi = 0 \ . \tag{5.29}$$

In the traveling wave solution $\phi = \phi(x - vt)$ which corresponds to a rotation in ϕ by 2π (as x goes from $-\infty$ to ∞) have the form,

$$\phi(x,t) = 4tan^{-1}[e^{\pm(\frac{x-vt}{\sqrt{1-v^2}})}] \ . \tag{5.30}$$

The $+$ sign is called soliton and the $-$ sign is called anti-soliton.

Now, we rewrite the lagrangian in Eq.(5.28) as follow:

$$\mathcal{L} = \frac{1}{2}\phi_t^2 - \frac{1}{2}\phi_x^2 + V(\phi) \ , \tag{5.31}$$

where $V(\phi) = cos\phi$. Using Taylor series, we can expand the potential $V(\phi)$ follow,

$$V(\phi) = \frac{1}{2}\phi^2 - \frac{1}{4!}\phi^4 + \cdots \ , \tag{5.32}$$

if we take into account up to second order. Here m is the mass and λ is the self-interaction coupling. The potential becomes,

$$V(\phi) = \frac{m^2}{2!}\phi^2 - \frac{\lambda}{4!}\phi^4 + \cdots \ . \tag{5.33}$$

Again, using the Euler-lagrange equation, we obtain,

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} - m^2\phi + \frac{\lambda}{3!}\phi^3 = 0 \ . \tag{5.34}$$

This equation is called the Sine-Gordon equation (approximation) or the nonlinear Klein-Gordon equation. The equation is a continuum version of the equation that describes a propagation of molecular vibration (vibron) in $\alpha-$helical protein. The structure of $\alpha-$helical protein chain can be seen in Fig. 5.1.

The vibration excitation in the $\alpha-$helix protein propagates from one group to the next because of the dipole-dipole interaction between the group. The wave is called the Davidov soliton [28]. Davydov has shown that in $\alpha-$helical protein soliton can be formed by coupling the propagation of amide$-I$ vibrations with longitudinal phonons along spines and that such entities are responsible for mechanism of energy transfer in biological system [28]. The similar wave also appears in the DNA molecules. An energitic solvent molecules (protein, drugs or some other ligands) kick DNA and create an elastic solitary wave [29]. The solitary waves (soliton) are described by the Sine-Gordon equation or the Nonlinear Schrodinger equation [30, 31].

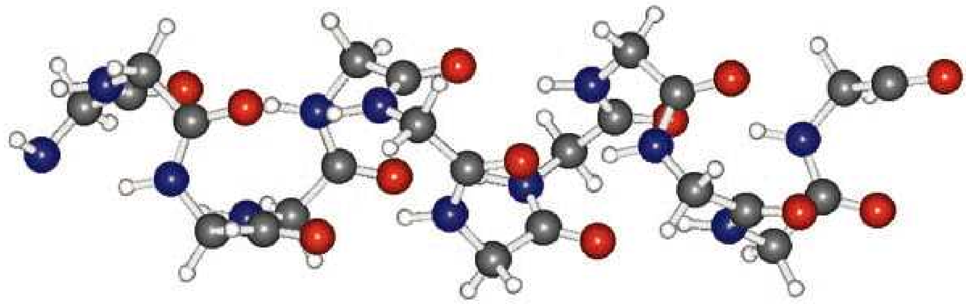

Figure 5.1: The atomic structure of α-helix in protein.

If $\alpha-$helical protein immersed in Bio-fluid, then the phenomenon can be described by the interaction of soliton with fluid system. Let us generalize the equation Eq. (5.34) into four vector formalism,

$$\partial_\mu \partial^\mu \phi - m^2 \phi + \frac{\lambda}{3} \phi^3 = 0 \ . \tag{5.35}$$

The equation has lagrangian density,

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) - \frac{m^2}{2!} \phi^2 + \frac{\lambda}{4!} \phi^4 \ . \tag{5.36}$$

We have developed that the fluid system can be describe by the lagrangian density,

$$\mathcal{L}_{NS} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \ . \tag{5.37}$$

Introducing the covariant derivative,

$$\mathcal{D}_\mu \phi = (\partial_\mu + ig\mathcal{A}_\mu)\phi \ , \tag{5.38}$$

we can apply the gauge field theory approach. The interaction between soliton and fluid system obeys the lagrangian,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} (\mathcal{D}_\mu \phi)(\mathcal{D}^\mu \phi) + \frac{m^2}{2} \phi^2 + \frac{\lambda}{4!} \phi^4 \ . \tag{5.39}$$

We are also able to write the Lagrangian as,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} (\mathcal{D}_\mu \phi)(\mathcal{D}^\mu \phi) + \frac{m^2}{2!} \phi^2 + V'(\phi) \ , \tag{5.40}$$

where $V'(\phi) = \frac{\lambda}{4} \phi^4$ can be interpreted as a self-interaction potential.

One interesting case is when we consider a static condition, $i.e. \partial_t f = 0$ with f is an arbitrary function. Substituting $\mathcal{A}_\mu = (\Phi, -\vec{v})$ and $V'(\phi) = \frac{\lambda}{4}\phi^4$, Eq. (5.40) becomes,

$$\mathcal{L} = -\frac{1}{2}(\nabla \times \vec{v})^2 + \frac{1}{2}|(\nabla - ig\vec{v})\phi|^2 + \frac{m^2}{2!}\phi^2 - \frac{\lambda}{4!}\phi^4 \ . \tag{5.41}$$

The lagrangian is similar with the Ginzburg-Landau free energy lagrangian that is widely used in superconductor theory [34].

We have seen that the phenomenon of $\alpha-$helical protein immersed in fluid similar with quantum electrodynamics for boson particle, while for static case it is similar with the Ginzburg-Landau model for superconductor. To get an explicit calculation, suppose we have one-dimensional velocity in x direction $\vec{v} = (u(x), 0, 0)$ and $\phi = \phi(x)$. Then the lagrangian in Eq.(5.41) reads,

$$\mathcal{L} = \frac{1}{2}\phi_x^2 - \frac{1}{2}g^2u^2\phi^2 + \frac{m^2}{2!}\phi^2 - \frac{\lambda}{4!}\phi^4 \ . \tag{5.42}$$

Substituting it into Euler-lagrangian equation we arrive at,

$$\frac{d^2\phi}{dx^2} + g^2u^2\phi - m^2\phi + \frac{\lambda}{12}\phi^3 = 0 \ . \tag{5.43}$$

We can write it in better form as,

$$\frac{d^2\phi}{dx^2} - \gamma(x)\phi + \frac{\lambda}{12}\phi^3 = 0 \ , \tag{5.44}$$

where $\gamma(x) = m^2 - g^2u(x)^2$. The equation is called the variable coefficient of non-linear Klein-Gordon equation.

To solve this equation, first consider the fluid velocity is constant, $i.e. u(x) = U$, then we have,

$$\frac{d^2\phi}{dx^2} - \gamma\phi + \alpha\phi^3 = 0 \ , \tag{5.45}$$

where $\gamma = m^2 - g^2U^2$ and $\alpha = \lambda/12$. The equation is similar with an-harmonics oscillation equation. The standard method to solve the equation is the perturbation method. In this thesis we will solve the equation without a perturbation methods. We use a mathematical trick as follows. First multiply it by $d\phi/dx$,

$$\frac{d\phi}{dx}\frac{d^2\phi}{dx^2} - \gamma\phi\frac{d\phi}{dx} + \alpha\phi^3\frac{d\phi}{dx} = 0 \ , \tag{5.46}$$

then rewriting the equation as,

$$\frac{1}{2}\frac{d}{dx}[\frac{d\phi}{dx}]^2 - \frac{\gamma}{2}\frac{d\phi^2}{dx} + \frac{\alpha}{4}\frac{d\phi^4}{dx} = 0 \ . \tag{5.47}$$

Integrating out this equation over x and putting the integration constant as zero due to integrable condition $\lim_{x\to\pm\infty}\phi = 0$, the equation becomes,

$$\left(\frac{d\phi}{dx}\right)^2 - \gamma\phi^2 + \frac{\alpha}{2}\phi^4 = 0 \ , \tag{5.48}$$

and it can be rewritten further as,

$$\int \frac{d\phi}{\phi(\delta^2 - \phi^2)^{\frac{1}{2}}} = \int \sqrt{\frac{\alpha}{2}} dx \ , \tag{5.49}$$

where $\delta^2 = \frac{2\gamma}{\alpha}$. The left hand side is,

$$-\frac{1}{\delta}\ln|\frac{\delta + \sqrt{\delta^2 - \phi^2}}{\phi}| = \sqrt{\frac{\alpha}{2}}x \ . \tag{5.50}$$

Solving the equation for ϕ we get,

$$\begin{aligned} \phi &= \frac{2\delta e^{-\sqrt{\frac{\alpha}{2}}\delta x}}{1 + e^{-2\sqrt{\frac{\alpha}{2}}\delta x}} = \frac{2\delta}{e^{\sqrt{\frac{\alpha}{2}}\delta x} + e^{-\sqrt{\frac{\alpha}{2}}\delta x}} \\ &= \frac{\delta}{\cosh(\sqrt{\frac{\alpha}{2}}\delta x)} = \delta \mathrm{sech}(\sqrt{\frac{\alpha}{2}}\delta x) \ . \end{aligned} \tag{5.51}$$

Thus, the solution for a homogeneous nonlinear Klein - Gordon equation is,

$$\phi(x) = A \mathrm{sech}(\Lambda x) \ , \tag{5.52}$$

where $A = \frac{24\gamma}{\lambda}$ and $\Lambda = \frac{24\sqrt{3}\gamma}{\lambda^{\frac{3}{2}}}$. This is depicted in Fig. 5.2.

The second application is ***the theory of turbulence***. The phenomenon of turbulence has been known for more than a hundred years but it remains to be one of the unsolved problem of modern physics. Its formulation is simple. The incompressible fluid is governed by two simple equation [1],

$$\begin{aligned} \rho\left(\frac{\partial\vec{v}}{\partial t} + (\vec{v}.\vec{\nabla})\vec{v}\right) &= -\vec{\nabla}P + \nu\nabla^2\vec{v} \ , \\ \vec{\nabla}\cdot\vec{v} &= 0 \ , \end{aligned} \tag{5.53}$$

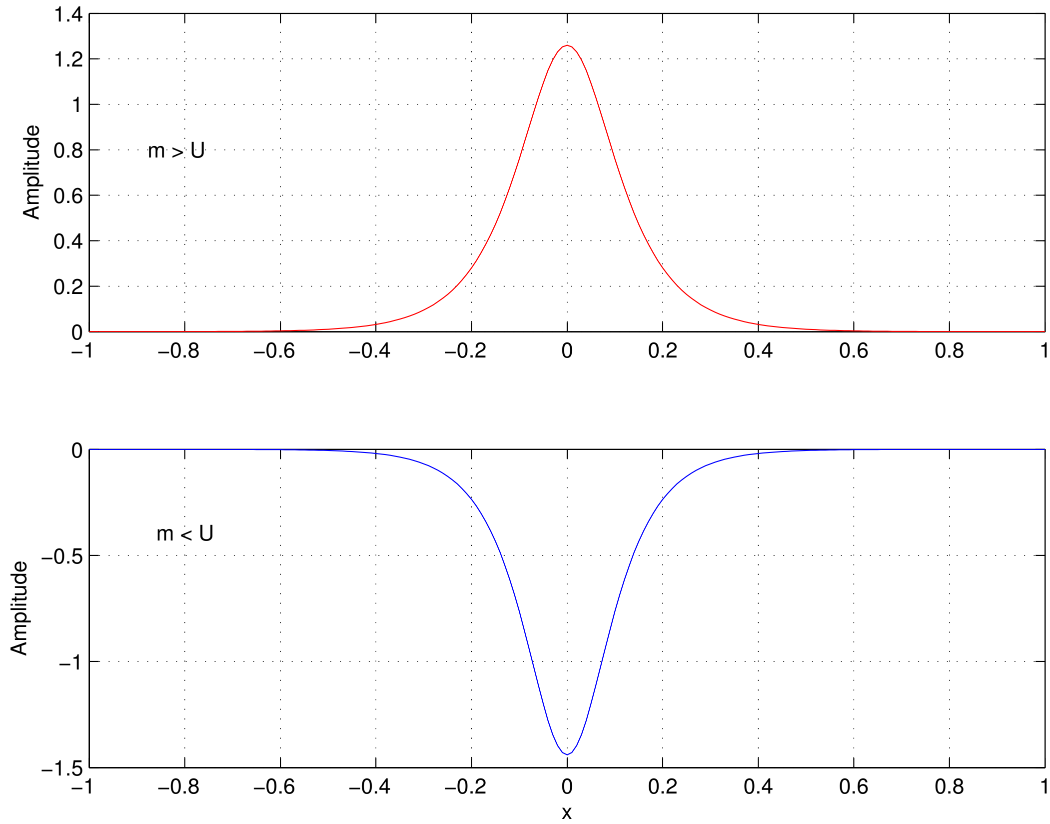

Figure 5.2: Single soliton solution of the nonlinear Klein-Gordon equation.

for the unknowns $\vec{v}$ and P (velocity and pressure). The problem is how to solve these equations with the appropriate boundary condition to find the motion of the fluid completely. But it is known for long time, it can be done for the laminar flow which describes the small velocity. As the velocity increases, the flow becomes unstable and the fluid swithches to a new regime of very complex motions with the velocity pulsating almost randomly and without any noticeable order. The important question is, what exactly is going on when the fluid is in such regime.

Recently the theory of turbulence lays on two main lines of research, $i.e.$the dynamical system approach and statistical theory. The dynamical system approach attempts to describe turbulence as deterministic chaos. The solution is usually described by a strange attractor in finite region phase space. This approach is based on simplified system of nonlinear evolution equations. Turbulence is expected to be a generic feature of such system [3]. The second approach is the statistical theory of turbulence. This approach treats the velocity field as a random variable and attempts to calculate correlation function [1, 32]. The final results is an hierarchy of equation relates these correlation functions. Due to close correlation between statistical mechanics and field theory, many researchers attempts to study turbulence using field theory. Our formulation is field theory, so we might be able to treats turbulence as a statistical theory approach. The turbulence flow is characterized by fluctuating velocity which is a random variable.

To formulate turbulence flow using field theory approach we make an analogy with statistical mechanics formalism. Consider a statistical mechanics system. The system whics microstate (for example spin in ferromagnetics) can be specified by N variable of spin denoted by $s_1, ..., s_N$. The dynamics of the system is usually studied in term of the correlation function that is defined as [34],

$$\langle s_i s_j \rangle = \frac{1}{\mathcal{Z}} \frac{\partial^2 \mathcal{Z}}{\partial J_i \partial J_j} \ , \tag{5.54}$$

where $\mathcal{Z} = e^{-\beta H}$ is a partition function and H is the Hamiltonian of the system. The famous model in the ferromagnetic research is the Ising model. J is related to the magnetic field $\overrightarrow{B}$ by $J = -\beta \overrightarrow{B}$. The thermal average of $s_{i1}...s_{in}$ is called the $n-$point correlation function of the spins, is written by [34],

$$\langle s_{i1}...s_{in} \rangle = \frac{1}{\mathcal{Z}} \frac{\partial^n \mathcal{Z}}{\partial J_{i1}...\partial J_{in}} \ . \tag{5.55}$$

The physical observable can be defined in term of correlation function. For example a generalized susceptibility related to the correlation function by equation [34],

$$\chi = \frac{\beta}{N} \Sigma_{ij} \langle s_i s_j \rangle \ . \tag{5.56}$$

In the turbulence flow, the random variable is fluid velocity u_i. Analogue to the ferromagnetics system the quantities such as $\langle u_i u_j \rangle$ are called correlation function. The average of $u_{i1}...u_{in}$ is called the $n-$point correlation function and is written,

$$\langle u_i(x_1) u_j(x_2)...u_l(x_n) \rangle = \frac{1}{i^n} \frac{\delta^n \mathcal{Z}}{\delta J_i(x_1) \delta J_j(x_2)...\delta J_l(x_n)} |_{J=0} \ . \tag{5.57}$$

In the field theory approach, the random variable $\langle u_i \rangle$ can be viewed as a field. The statistical partition function exhibit a close analogy to the generating functional of field theory. Using this analogy, if $u(x)$ is a field then the $n-$point correlation function can be defined as,

$$\langle u(x_1)u(x_2)...u(x_n)\rangle = \frac{1}{i^n}\frac{\delta^n \mathcal{Z}}{\delta J(x_1)\delta J(x_2)...\delta J(x_n)}|_{J=0} \ , \tag{5.58}$$

when the generating functional is given by [6],

$$\mathcal{Z} = \int \mathcal{D}(u)e^{\int \mathcal{L}dx} \ . \tag{5.59}$$

This equation is a conventional quantum field theory approach to turbulence [13, 14]. Recently, using this equation, perturbation field-theoretic techniques have been useful in turbulence research. The method is called the renormalization group methods (RG). The RG approach to the Stochastic Navier-Stokes equation allows us to prove the existence of the infrared scale invariance with exactly known Kolmogorov dimension and calculate of representation constant such as the Kolmogorov constant in a reasonable agreement with experiment [10]. The RG approach to turbulent flow is based on the generating functional. Using standard procedure in field theory the large distance long time behaviour, effective eddy viscosity, turbulence cascade and the other transport coefficient can be investigated [11, 12, 13, 14].

In our formalism, the random variable is denoted by $\mathcal{A}_\mu$ so that the $n-$point correlation function is,

$$\langle \mathcal{A}_\mu(x_1)\mathcal{A}_\nu(x_2)...\mathcal{A}_\sigma(x_n)\rangle = \frac{1}{i^n}\frac{\delta^n \mathcal{Z}[J]}{\delta J(x_1)\delta J(x_2)...\delta J(x_n)}|_{J=0} \ , \tag{5.60}$$

and the generating functional is given by,

$$\mathcal{Z} = \int \mathcal{D}\mathcal{A}_\mu e^{\int [\mathcal{L}_{NS}+J_\mu \mathcal{A}^\mu]dx^\mu} \ . \tag{5.61}$$

J is a source, where $\mathcal{L}_{NS} = -\frac{1}{4}\mathcal{F}_{\mu\nu}F^{\mu\nu} = \frac{1}{2}\mathcal{A}^\mu[g_{\mu\nu}\Box - \partial_\mu\partial_\nu]\mathcal{A}^\nu$ for 'free' field. The expansion of the generating functional $\mathcal{Z}$ in the perturbation series generates the Feynman diagram techniques.

Some physical observable can be calculated by the correlation function. For example the pair correlation of the velocity (in momentum space) associated with a physical observable given by [33],

$$\langle u_\alpha((\vec{k},t)u_\alpha(\vec{k}',t')\rangle = \frac{E(k)}{4\pi k^2}(\delta_{\alpha\beta} + \frac{k_\alpha k_\beta}{k^2})\delta(\vec{k}-\vec{k}') \ . \tag{5.62}$$

The energy spectrum E(k) is defined via the kinetic energy dissipation rate as,

$$\epsilon = \int_o^\infty 2\nu_o k^2 E(k) dk \ . \tag{5.63}$$

ν_o is the kinematic viscosity and the kinetic energy dissipation (ϵ) can be determined by experiment or by observation [9, 32].

Turbulence has a scaling laws. Similar with critical phenomena, the theory of the renormalized group can be used to find the scaling laws of turbulence [12, 13, 14].

Chapter 6

Conclusion

The true law cannot be linear nor can they be derived from such
Albert Einstein

We have shown that there are similarity between electromagnetics field and fluid dynamics using the Maxwell-like equation for an ideal fluid. These results provide a clue that we might be able to build a lagrangian density using bosonic lagrangian (abelian gauge theory) which is called the Navier-Stokes lagrangian in term of scalr and vector potentials $\mathcal{A}_\mu$. Then the Navier-Stokes equation is obtained as its equation of motion through the Euler-lagrange principle. We have obtained the same results for both Minkowski and Euclidean space-time formulations. The application of the theory is wide, for instance the interaction between Davydov soliton with fluid system that can be described by the lagrangian density which is similar to quantum electrodynamics for boson particle. In the static condition, the lagrangian density is similar with the Ginzburg-Landau lagrangian. If the fluid flow is parallel with soliton propagation we also obtain the variable coefficient Nonlinear Klein-Gordon equation. Single soliton solution has been obtained in term of a second hyperbolic function. Using similarities between the statistical mechanics and the fields theory we can construct the theory of turbulence. The $n-$point correlation function is describe by the generating functional that similar with quantum electrodynamics.

More detail calculation on the application of our approach into some real phenomenon, for instance turbulence and nano-crystal, can be seen in separate works [35].

Appendix
element of tensor analysis

In a non Euclidean vector space, the space-time continuous is defined in terms of a four dimensional space with coordinate x^0, x^1, x^2, x^3. There is a well-defined transformation that yields new coordinates $x^{'0}, x^{'1}, x^{'2}, x^{'3}$ according to the role:

$$x^\alpha = x'^\alpha(x^0, x^1, x^2, x^3) \tag{6.1}$$

The transformation law is not specified. Tensor of rank k associated with the space-time point x are defined by their transformation properties under the transformation $x \rightarrow$. For example, tensor of rank zero (a scalar) is a single quantity whose value is not changed by the transformation. Tensors of rank one (a vector) have two kinds of vectors. The first is called a contravariant (A^α) that are transformed according to the rule:

$$A^\alpha \rightarrow A'^\alpha = \frac{\partial x'^\alpha}{\partial x^\beta} A^\beta = \frac{\partial x'^\alpha}{\partial x^0} A^0 + \frac{\partial x'^\alpha}{\partial x^1} A^1 + \frac{\partial x'^\alpha}{\partial x^2} A^2 + \frac{\partial x'^\alpha}{\partial x^3} A^3 \tag{6.2}$$

The summation convention just for repeated indices. A covariant vector (A_α) is defined by the rule:

$$A_\alpha \rightarrow A'_\alpha = \frac{\partial x^\beta}{\partial x'^\alpha} A_\beta = \frac{\partial x'^0}{\partial x^\alpha} A_0 + \frac{\partial x'^1}{\partial x^\alpha} A_1 + \frac{\partial x'^2}{\partial x^\alpha} A_2 + \frac{\partial x'^3}{\partial x^\alpha} A_3 \tag{6.3}$$

The contravariant tensor of rank two $F^{\alpha\beta}$ (consists of 16 quantities) that transform according to the rule:

$$F^{\alpha\beta} \rightarrow A'^{\alpha\beta} = \frac{\partial x'^\alpha}{\partial x^\gamma} \frac{\partial x'^\beta}{\partial x^\delta} F^{\gamma\delta} \tag{6.4}$$

A covariant tensor of rank two $F_{\alpha\beta}$ transform according to:

$$F_{\alpha\beta} \rightarrow A'_{\alpha\beta} = \frac{\partial x^\gamma}{\partial x'^\alpha} \frac{\partial x^\delta}{\partial x'^\beta} F_{\gamma\delta} \tag{6.5}$$

Define a tensor of rank two as follow:

$$\mathcal{F}_{\mu\nu} \equiv \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu \tag{6.6}$$

This is an anti-symmetry tensor of rank two.

Prove:

$$\begin{aligned}
\mathcal{F}_{\mu\nu} \to \mathcal{F}'_{\mu\nu} &= \partial_\mu \mathcal{A}'_\nu - \partial_\nu \mathcal{A}'_\mu \\
&= \frac{\partial^2 x^\sigma}{\partial x'^\mu \partial' x^\nu} A_\sigma + \frac{\partial x^\sigma}{\partial x'^\mu}\frac{\partial A_\sigma}{\partial x'^\nu} - \frac{\partial^2 x^\sigma}{\partial x'^\mu \partial x'^\nu} A_\sigma - \frac{\partial x^\sigma}{\partial x'^\nu}\frac{\partial A_\sigma}{\partial x'^\mu} \\
&= \frac{\partial x^\sigma}{\partial x'^\mu}\frac{\partial A_\sigma}{\partial x'^\nu} - \frac{\partial x^\sigma}{\partial x'^\nu}\frac{\partial A_\sigma}{\partial x'^\mu} \\
&= \frac{\partial x^\sigma}{\partial x'^\mu}\frac{\partial x^\alpha}{\partial x'^\nu}\frac{\partial A_\sigma}{\partial x^\alpha} - \frac{\partial x^\sigma}{\partial x'^\nu}\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial A_\sigma}{\partial x^\alpha} \\
&= \frac{\partial x^\sigma}{\partial x'^\mu}\frac{\partial x^\alpha}{\partial x'^\nu}\frac{\partial A_\sigma}{\partial x^\alpha} - \frac{\partial x^\alpha}{\partial x'^\nu}\frac{\partial x^\sigma}{\partial x'^\mu}\frac{\partial A_\alpha}{\partial x^\sigma} \\
&= \frac{\partial x^\sigma}{\partial x'^\mu}\frac{\partial x^\alpha}{\partial x'^\nu}[\frac{\partial A_\sigma}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\sigma}] \\
&= \frac{\partial x^\sigma}{\partial x'^\mu}\frac{\partial x^\alpha}{\partial x'^\nu}\mathcal{F}_{\sigma\alpha}
\end{aligned} \tag{6.7}$$

The norm or metric is a special case of the general differential length element,

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta \tag{6.8}$$

where $g_{\alpha\beta} = g_{\beta\alpha}$ is called the metric tensor. The Minkowsky space-time is defined by:

$$g_{00} = 1, g_{11} = g_{22} = g_{33} = -1 g_{\alpha\beta} = 0 \Rightarrow \alpha \neq \beta \tag{6.9}$$

For the flat space-time we have $g_{\alpha\beta} = g^{\alpha\beta}$ and $g_{\alpha\beta} g^{\alpha\beta} = \delta^\beta_\alpha$ is the kronecker delta. The differential operator is defined by $\partial_\alpha = \frac{\partial}{\partial x^\alpha} = (\partial_0, -\vec{\nabla})$. Using the metric tensor $g_{\alpha\beta}$ we have useful formulae:

$$\begin{aligned}
\frac{\partial A^\alpha}{\partial A^\beta} &= \delta^\alpha_\beta \\
\frac{\partial A_\alpha}{\partial A_\beta} &= \delta^\alpha_\beta
\end{aligned} \tag{6.10}$$

$$\frac{\partial A_\alpha}{\partial A^\beta} = \frac{\partial}{\partial A^\beta} g_{\alpha\gamma} A^\gamma = g_{\alpha\gamma}\frac{\partial A^\gamma}{\partial A^\beta} = g_{\alpha\gamma}\delta^\beta_\gamma = g_{\alpha\beta} \tag{6.11}$$

$$\frac{\partial A^\alpha}{\partial A_\beta} = \frac{\partial}{\partial A_\beta} g^{\alpha\gamma} A_\gamma = g^{\alpha\gamma} \frac{\partial A_\gamma}{\partial A_\beta} = g^{\alpha\gamma} \delta_\gamma^\beta = g^{\alpha\beta} \tag{6.12}$$

$$\frac{\partial F^{\alpha\beta}}{\partial F^{\mu\nu}} = \delta_\mu^\alpha \delta_\nu^\beta - \delta_\nu^\alpha \delta_\mu^\beta \tag{6.13}$$

$$\begin{aligned} \frac{\partial F_{\alpha\beta}}{\partial F^{\mu\nu}} &= \frac{\partial}{\partial F^{\mu\nu}} g_{\alpha\gamma} g_{\beta\delta} F^{\gamma\delta} = g_{\alpha\gamma} g_{\beta\delta} \frac{\partial F^{\gamma\delta}}{\partial F^{\mu\nu}} \\ &= g_{\alpha\gamma} g_{\beta\delta} [\delta_\mu^\gamma \delta_\nu^\delta - \delta_\nu^\gamma \delta_\mu^\delta] = g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\nu} g_{\beta\mu} \end{aligned} \tag{6.14}$$

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